AD616643

AMRL-TR-65-40

RESPONSE OF A LINEAR DAMPED DYNAMIC SYSTEM TO SELECTED ACCELERATION INPUTS

STANLEY BARRETT
PETER R. PAYNE

FROST ENGINEERING DEVELOPMENT CORPORATION

APRIL 1965

D D C

JUN 2 2 1965

DDC-IRA   E

NOTICES

When US Government drawings, specifications, or other data are used for any purpose other than a definitely related Government procurement operation, the Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise, as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Requests for copies of this report should be directed to either of the addressees listed below, as applicable:

Federal Government agencies and their contractors registered with Defense Documentation Center (DDC):

DDC
Cameron Station
Alexandria, Virginia 22314

Non-DDC users (stock quantities are available for sale from):

Chief, Input Section
Clearinghouse for Federal Scientific & Technical Information (CFSTI)
Sills Building
5285 Port Royal Road
Springfield, Virginia 22151

Change of Address

Organizations and individuals receiving reports via the Aerospace Medical Research Laboratories automatic mailing lists should submit the addressograph plate stamp on the report envelope or refer to the code number when corresponding about change of address or cancellation.

Do not return this copy. Retain or destroy.

RESPONSE OF A LINEAR DAMPED DYNAMIC SYSTEM TO SELECTED ACCELERATION INPUTS

STANLEY BARRETT
PETER R. PAYNE

FOREWORD

This study was initiated by the Biophysics Laboratory of the Aerospace Medical Research Laboratories, Aerospace Medical Division, Wright-Patterson Air Force Base, Ohio, with the support of the Biodynamics Section, Environmental Physiology Branch, Life Systems Division, National Aeronautical and Space Administration, Manned Spacecraft Center, Houston, Texas. The research was conducted by Frost Engineering Development Corporation, 3910 South Kalamath Street, Englewood, Colorado, under Contract No. AF33(657)-9514. Mr. Peter R. Payne was the principal investigator. Mr. James Brinkley of the Vibration and Impact Branch was the Contract Monitor for the Aerospace Medical Research Laboratories, while Mr. Harris F. Scherer was the NASA liaison representative.

The information presented in this report is a summary and extension of the analytical solutions for the response of a dynamic system used to represent the mechanical response characteristics of the human body. The solutions described within this report were required to supplement a research program concerned with the investigation of human body support and restraint system dynamics.

The research presented in this report was initiated in April 1963 and completed in June 1963.

This report is catalogued by Frost Engineering Development Corporation as Technical Report 194-10 and is one of a series of reports generated in the area of human restraint and support system dynamics.

This technical report has been reviewed and is approved.

J. W. HEIM, PhD
Technical Director
Biophysics Laboratory

ii

ABSTRACT

The general theory is developed for the response of a single degree of freedom dynamic system to an arbitrary acceleration forcing function. Closed-form solutions are obtained for a variety of discrete pulse shapes using the method of Laplace transforms and the form of the solutions indicated for oscillatory inputs and semi-infinite ramps, in terms of complex Fourier series. A comparison of base and mass excitation of the system is included. In previously published work on this subject, analytical solutions are in general only given for undamped systems; an exception is the response to a sine-wave, which appears in many standard texts. The dynamic analysis of the human body usually considers models involving damping, so that in this area there is a definite need for the extensions given.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

a time parameter, used to define pulse duration

A substitution for $(\psi_2 - \gamma_3)$

$A(n)$ Fourier transform of impulsive response function

B substitution for $(\psi_2 - \gamma_3 - \eta\frac{\pi}{2})$

c linear damping coefficient

$\bar{z}$ linear damping coefficient ratio, $\bar{z} = \frac{c}{\omega}$

C_n nth coefficient in the complex Fourier series

D substitution for $\frac{2\pi}{a}$

$f(x)$ function of x

$f_1(\ddot{y}_c)$ function of input to system

$f_2(\delta)$ function of output of system

$F_1(p)$ Laplace transform of $f_1(\ddot{y}_c)$

$F_2(p)$ Laplace transform of $f_2(\delta)$

$g(\tau)$ impulsive response function; also, inverse Laplace transform of $G(p)$

$G(p)$ system transfer function

$H(\tau)$ Heaviside unit function,

$$H(\tau) = 1 \text{ for } \tau > 1$$
$$= 0 \text{ for } \tau < 1$$

i $\sqrt{-1}$

J_1 substitution for $\sqrt{1 - \dfrac{D^2}{\{(D^2-1)^2 + 4\bar{z}^2 D^2\}^{1/2}} \cos(\psi_1 - \gamma_3)}$

J_2 substitution for $\sqrt{1 + \dfrac{D^2}{\{(D^2-1)^2 + 4\bar{z}^2 D^2\}^{1/2}} \cos(\psi_1 - \gamma_3)}$

k linear spring rate

K linear damping force coefficient, $K = cm$

$\mathcal{L}(x)$ Laplace transform of function $x(\tau)$

$\mathcal{L}^{-1}(f)$ inverse Laplace transform of function $f(p)$

m mass of basic system

max	subscript indicating maximum value
n	arbitrary integer
o	subscript, indicating initial condition
p	complex quantity, occurring in definition of Laplace transform
s	arbitrary variable of integration
t	time
T	period of oscillatory acceleration input
v	velocity
x	arbitrary function, as used in definition of $\mathcal{L}(x)$
y_m	vertical ordinate of point mass
$\ddot{y}_c$	input acceleration, applied to base of system as function of real time
y_c''	input acceleration as function of non-dimensional time
$\ddot{Y}_c$	peak value of $\ddot{y}_c$
Y_c''	peak value of y_c''
δ	deflection of spring
$\delta(t)$	conventional symbol for Dirac delta function
Δ	symbol indicating incremental quantity, as in Δv
η	substitution for $\sqrt{1-\zeta^2}$
$\theta, \theta_1, \theta_2$	phase angles
λ	length of spring before application of input
ζ	substitution for $\dfrac{\omega^2 \delta_0}{\omega \, \Delta v}$
τ	non-dimensional time parameter, $\tau = \omega t$
τ^*	value of τ corresponding to maximum deflection of system
φ	phase angle
ψ_1, ψ_2, ψ_3	phase angles
ω	undamped natural frequency of a linear system, $\omega = \sqrt{\dfrac{k}{m}}$
Ω	frequency of oscillation of input

Dots above symbol indicate differentiation with respect to real time,

$$\text{i.e.,} \quad \ddot{y}_c = \frac{d^2}{dt^2}(y_c)$$

Primes after symbol indicate differentiation with respect to non-dimensional time,

$$\text{i.e.,} \quad y_c'' = \frac{d^2}{d\tau^2}(y_c)$$

SECTION 1

INTRODUCTION

This report is concerned with the behavior of a spring/mass/damper system of the type shown in Figure 1. This model has been used in the past to simulate the dynamic response of parts of the human body, in efforts to assess human tolerance to acceleration.

The response of this system to a continuous sinusoidal input is frequently treated in engineering texts; however, the solution for other types of input is usually confined to an undamped system. Mindlin, et al (Ref. 1) have presented results showing the response to a half-cycle sine pulse. Although an analytical solution was given for the zero-damping case, the curves for non-zero damping were plotted from analog computer results.

In Ref. 2 it is pointed out that in the absence of damping the base excitation can be expressed in terms of displacement or velocity as alternatives to acceleration. Although this is also possible when damping exists, the resulting equations contain more than one input derivative and are generally too complicated to be considered for engineering applications.

We shall therefore confine our attention to the response of the system when the base of the spring is accelerated, but it is shown that the results are applicable, by a simple substitution, to the situation in which the mass is excited while the spring/damper element is grounded.

SECTION 2

THE GENERAL THEORY OF A LINEAR SYSTEM

2.1   Comparison of Base and Mass Excitations:

Consider the basic system (Fig. 1) with the excitation $\ddot{y}_c = \ddot{y}_c(t)$ applied at the base.   Then, if

$\ddot{y}_m$ = acceleration of the mass,

$k\delta$ = force in the spring, $\Big\}$ (1)

$2K\dot{\delta}$ = force on the damper,

we have

$$m\ddot{y}_m = k\delta + 2K\dot{\delta}$$

or $\ddot{y}_m = \omega^2\delta + 2c\dot{\delta}$ (2)

since $\omega = \sqrt{\dfrac{k}{m}}$
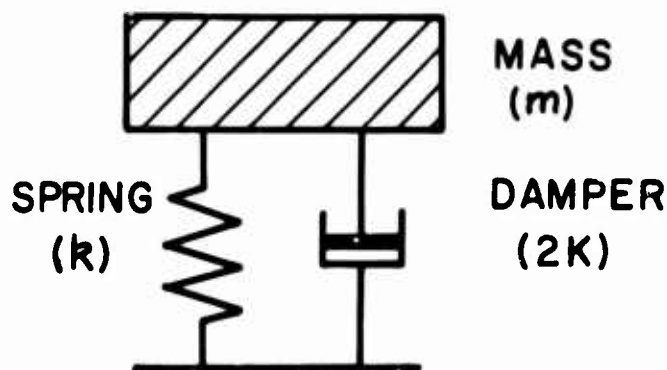
and $c = \dfrac{K}{m}$

Figure 1. Single Degree of Freedom System.

Let λ = length of spring when unloaded

and δ = deflection of spring

so that $\delta = \lambda - (y_m - y_c)$

$$\dot{\delta} = \quad -\dot{y}_m + \dot{y}_c$$

$$\ddot{\delta} = \quad -\ddot{y}_m + \ddot{y}_c$$

$$\therefore \quad \ddot{y}_m = \quad -\ddot{\delta} + \ddot{y}_c \tag{3}$$

Substituting eq. (3) into eq. (2) and rearranging gives

$$\ddot{y}_c = \quad \ddot{\delta} + 2c\dot{\delta} + \omega^2\delta \tag{4}$$

Consider now the case in which the mass experiences a force excitation $P = P(t)$, with the base of the system fixed.

Corresponding to eq. (2) we have, in general,

$$\ddot{y}_m = \frac{P}{m} + \omega^2 \delta + 2c\dot{\delta} \tag{5}$$

Since the base is fixed,

$$y_c = \text{constant},$$

and

$$\dot{y}_c = \ddot{y}_c = 0$$

eq. (3) becomes

$$\ddot{y}_m = -\ddot{\delta}$$

and substitution into eq. (5) leads to

$$-\frac{P}{m} = \ddot{\delta} + 2c\dot{\delta} + \omega^2\delta \tag{6}$$

This indicates that solutions obtained for the base excitation configuration can be applied to mass excitation problems simply by substituting $(-P/m)$ for $\ddot{y}_c$.

2.2   General Solution:

For a linear system, we have seen that

$$\ddot{\delta} + 2c\dot{\delta} + \omega^2\delta = \ddot{y}_c(t) \tag{7}$$

Using the transform $\tau = \omega t$, eq. (7) becomes

$$\delta'' + 2\bar{c}\delta' + \delta = y_c''(\tau) \tag{8}$$

where the prime is used to indicate differentiation with respect to the non-dimensional time parameter τ, and $\bar{c} = \frac{c}{\omega}$

This equation is most conveniently solved for specific cases by the use of the operational calculus. The Laplacian transform is defined by

$$\mathcal{L}(x) = \int_0^\infty e^{-p\tau} x(\tau)\, d\tau \tag{9}$$

using the symbol $\mathcal{L}(x)$ to denote the transform of the function $x(\tau)$. The existence of the transform must be verified when the function to be transformed is not common enough to be tabulated. Three sufficiency conditions are available; these, if satisfied, guarantee the existence of $\mathcal{L}(x)$:

4

(a) $x(\tau)$ is at least piecewise continuous in any interval $\tau_1 < \tau < \tau_2$, where $\tau_1 > 0$

(b) $\tau^n |x(\tau)|$ is bounded near $\tau = 0$ for some $n < 1$.

(c) $e^{N\tau} |x(\tau)|$ is bounded for large values of τ, for some number N.

In Laplacian notation, eq. (8) becomes

$$\mathcal{L}(\delta)\left[p^2 + 2\bar{c}p + 1 \right] = \mathcal{L}(y_c'') + \delta_0(p + 2\bar{c}) + (\delta')_0 \quad (10)$$

where $\quad \delta_0$ = initial deflection at $\tau = 0$

and $\quad (\delta')_0$ = initial velocity at $\tau = 0$.

We shall concern ourselves only with subcritical damping, where $\bar{c} < 1.0$. Thus we can employ the substitution

$$\eta^2 = 1 - \bar{c}^2 \quad (11)$$

and write $\quad p^2 + 2\bar{c}p + 1 = p^2 + 2\bar{c}p + \bar{c}^2 + (1 - \bar{c}^2)$

$$= (p + \bar{c})^2 + \eta^2 \quad (12)$$

so that eq. (10) yields

$$\mathcal{L}(\delta) = \frac{\mathcal{L}(y_c'')}{(p+\bar{c})^2 + \eta^2} + \frac{\delta_0(p + 2\bar{c})}{(p+\bar{c})^2 + \eta^2} + \frac{(\delta')_0}{(p+\bar{c})^2 + \eta^2}$$

Taking the inverse transformation and making use of the linearity of the operator,

$$\delta(\tau) = \mathcal{L}^{-1}\left[\frac{\mathcal{L}(y_c'')}{(p+\bar{c})^2 + \eta^2}\right] + \delta_0 \mathcal{L}^{-1}\left[\frac{p + 2\bar{c}}{(p+\bar{c})^2 + \eta^2}\right]$$

$$+ (\delta')_0 \mathcal{L}^{-1}\left[\frac{1}{(p+\bar{c})^2 + \eta^2}\right] \quad (13)$$

In the second term of eq. (13),

$$\frac{p + 2\bar{c}}{(p+\bar{c})^2 + \eta^2} = \frac{(p+\bar{c}) + \bar{c}}{(p+\bar{c})^2 + \eta^2} \quad (14)$$

5

- giving, after inversion,

$$\delta_0 e^{-\bar{\epsilon}\tau}\left\{\cos \eta\tau + \frac{\bar{\epsilon}}{\eta}\sin \eta\tau\right\}$$

$$= \frac{\delta_0}{\eta} e^{-\bar{\epsilon}\tau}\sin(\eta\tau + \varphi) \tag{15}$$

$$\text{where} \quad \varphi = \arcsin \eta \tag{16}$$

For the initial velocity term,

$$(\delta')_0 \mathcal{L}^{-1}\left[\frac{1}{(p+\bar{\epsilon})^2 + \eta^2}\right] = \frac{(\delta')_0}{\eta} e^{-\bar{\epsilon}\tau}\sin \eta\tau \tag{17}$$

Thus the solution to eq. (13) may be written as

$$\delta = \mathcal{L}^{-1}\left[\frac{\mathcal{L}(y_c'')}{(p+\bar{\epsilon})^2 + \eta^2}\right] + \frac{e^{-\bar{\epsilon}\tau}}{\eta}\left[\delta_0 \sin \eta\tau + (\delta')_0 \sin(\eta\tau + \varphi)\right] \tag{18}$$

When the motion due to an initial condition only is required, this can be obtained immediately from eq. (18). Moreover, for any forcing function, the initial condition transients are directly additive to the response due to the forcing function. Thus we need only consider the variation in eq. (18) due to the particular forcing function $y_c''(\tau)$ in what follows.

We shall see in Section 3 that any acceleration input pulse or vibration has a characteristic peak value $Y_c'' = \frac{1}{\omega^2}\left(\text{max.}\left[\ddot{y}_c\right]\right)$

It is convenient to express δ in terms of Y_c'', so that eq. (18) becomes

$$\frac{\delta}{Y_c''} = \mathcal{L}^{-1}\left[\frac{\mathcal{L}(y_c'')\frac{1}{Y_c''}}{(p+\bar{\epsilon})^2 + \eta^2}\right] + \text{initial conditions solution.} \tag{19}$$

Initial conditions solution:

$$\frac{\delta}{Y_c''} = \frac{e^{-\bar{\epsilon}\tau}}{\eta Y_c''}\left[\delta_0 \sin \eta\tau + (\delta')_0 \sin(\eta\tau + \varphi)\right]$$

Consider the steady-state part of the solution given in eq. (19):

$$\frac{\delta}{Y_c''} = \mathcal{L}^{-1}\left[\frac{\mathcal{L}(y_c'')\frac{1}{Y_c''}}{(p+\bar{\epsilon})^2 + \eta^2}\right] = \frac{1}{Y_c''}\mathcal{L}^{-1}\left[\frac{\mathcal{L}(y_c'')}{(p+\bar{\epsilon})^2 + \eta^2}\right] \tag{20}$$

The Laplace transform of eq. (20) is simply

$$\mathcal{L}(\delta) = \frac{\mathcal{L}(y_c'')}{(p+\bar{\epsilon})^2 + \eta^2} \tag{21}$$

6

so that

$$\frac{1}{(p+\zeta)^2 + \eta^2} = \frac{\mathcal{L}(\delta)}{\mathcal{L}(y_c'')} \tag{22}$$

$\mathcal{L}(\delta)$ may be regarded as the transform of the "output" of the system, with $\mathcal{L}(y_c'')$ the transform of the "input"; thus, subject to quiescent initial conditions, we may define

$$G(p) = \frac{1}{(p+\zeta)^2 + \eta^2} = \frac{\mathcal{L}(\delta)}{\mathcal{L}(y_c'')} = \frac{F_2(p)}{F_1(p)} \tag{23}$$

- as the system transfer function for the input y_c'', with the input applied to the base of the system as shown in Fig. 1. It should be noted that definition (23) is only unique when the point of application of the input is specified.

The transfer function of the system is the Laplace transform of its rheonomic normal response (i.e., its response as an explicit function of time, for quiescent initial conditions) to a first order unit impulse.

From eq. (23), it follows that

$$F_2(p) = G(p) F_1(p) \tag{24}$$

Let

$$f_2(\delta) = \mathcal{L}^{-1}[F_2(p)]$$
$$= \mathcal{L}^{-1}[G(p) F_1(p)] \tag{25}$$

Now the inverse transform of the product of two transforms can be obtained by means of the convolution theorem:

$$\left. \begin{array}{l} \mathcal{L}^{-1}[G(p) F_1(p)] = \int_0^\tau g(\tau - s) f_1(s)\, ds \\[2ex] = \int_0^\tau g(s) f_1(\tau - s)\, ds \end{array} \right\} \tag{26}$$

where

$$G(p) = \mathcal{L}[g(\tau)]$$
$$= \frac{F_2(p)}{F_1(p)}$$

$$F_1(p) = \mathcal{L}(y_c'') = \mathcal{L}(f_1)$$
$$F_2(p) = \mathcal{L}(\delta) = \mathcal{L}(f_2)$$

and s is an arbitrary "dummy" variable of integration.

Thus, when an acceleration input function y_c'' is given, the response of a linear system to y_c'' may be found by integration of one of equations (26), making use of the known response of the system to a unit impulse. Although the technique is based on the Laplace transform approach, it is not necessary to use the transform of the input acceleration function. This method od is probably more useful than the direct inverse transformation procedure for cases where the transformation $\mathcal{L}(y_c'')$ is not a rational function of p, so that the ratio $\mathcal{L}(y_c'')/(p+z)^2+\eta$ cannot be expanded into simple partial fractions. In many cases, however, the integrand is so complicated that the integration becomes very laborious to perform analytically, and numerical methods are preferable.

The approach just discussed, involving the convolution integral, may be developed even further for the special case of a steady-state oscillatory input. The input may then be expressed as a complex Fourier series:

$$y_c'' = f_1(\tau) = \sum_{n=-\infty}^{\infty} c_n e^{in\Omega\tau} \tag{27}$$

Where c_n is the nth complex coefficient in the series, and Ω is the frequency of oscillation of the input. c_n is given by

$$c_n = \frac{\Omega}{2\pi} \int_{-\pi/\Omega}^{\pi/\Omega} f_1(\tau) e^{-in\Omega\tau} d\tau \tag{28}$$

The complex exponential form of the Fourier series thus needs only one formula for all its coefficients, and is more compact and easier to manipulate than the more common trigonometrical form. It also provides a convenient transition to Fourier integrals and Fourier transforms.

Re-writing eq. (27) for $f_1(\tau-s)$ and substituting into the second of equations (26) leads to

$$f_2 = \delta = \int_0^\infty g(s) \sum_{n=-\infty}^{\infty} c_n e^{in\Omega(\tau-s)} \tag{29}$$

$$= \sum_{-\infty}^{\infty} c_n e^{in\Omega\tau} \int_0^\infty g(s) e^{-in\Omega s} ds \tag{30}$$

since $c_n e^{in\Omega\tau}$ is independent of s

For convenience, let

$$\int_0^\infty g(s) e^{-in\Omega s} ds = A(n) \tag{31}$$

where $A(n)$ is thus the Fourier transform of the impulsive response function $g(\tau)$. Since $g(\tau)$ is zero for $\tau < 0$, the limits of integration can be $0, \infty$ or $\pm\infty$ without affecting the value of the integral.

Hence we can write (30) more succinctly as

$$\delta = \sum_{n=-\infty}^{\infty} A(n) c_n e^{in\Omega\tau} \tag{32}$$

For an input which is real, the output must also be real; under these conditions eq. (32) becomes

$$\delta = c_0 A(0) + 2\sum_{n=1}^{\infty}\left|c_n A(n)\cos\left(n\Omega\tau + \beta_n\right)\right| \tag{33}$$

where β_n is a phase angle, defined by

$$\tan\beta_n = \frac{-\int_{-\pi/\Omega}^{\pi/\Omega} f_1(\tau)\sin n\Omega\tau\, d\tau}{\int_{-\pi/\Omega}^{\pi/\Omega} f_1(\tau)\cos n\Omega\tau\, d\tau} \tag{34}$$

Since any periodic function can be expressed as a Fourier series to a high degree of accuracy, the response of a single degree of freedom system to any periodic function can be calculated from eq. (33). Fourier series representations of some periodic functions are given in Table 2 of Section 3.

It should be noted that the Fourier series expansion of a periodic function which has finite discontinuities leads to finite (though not usually significant) amplitude overshoots in the region of the discontinuities. For example, for a rectangular wave, the Fourier series expansion gives amplitude overshoots at the discontinuities which approach a limit of about $\pm$ 9% of the amplitude of the wave as the number of terms in the expansion tends to infinity (see Fig. 2). At points on the rectangular wave between the discontinuities the difference is much less, of course, decreasing as the number of terms taken in the expansion is increased. This behavior is due to the fact that the Fourier series expansion fails to converge uniformly at the discontinuities; this is known as the Gibbs phenomenon. It is discussed in detail in Ref. 3.

Input functions which are nonperiodic (e.g., acceleration pulses) may also be treated by a modification of the Fourier series expansion, in the form of Fourier integrals and Fourier transforms. Generally speaking, however, the method of Laplace transforms usually proves to be more convenient to use for such functions.

Further discussion of the Fourier methods is beyond the scope of this report. A more detailed treatment will be found in, for example, Ref. 3.
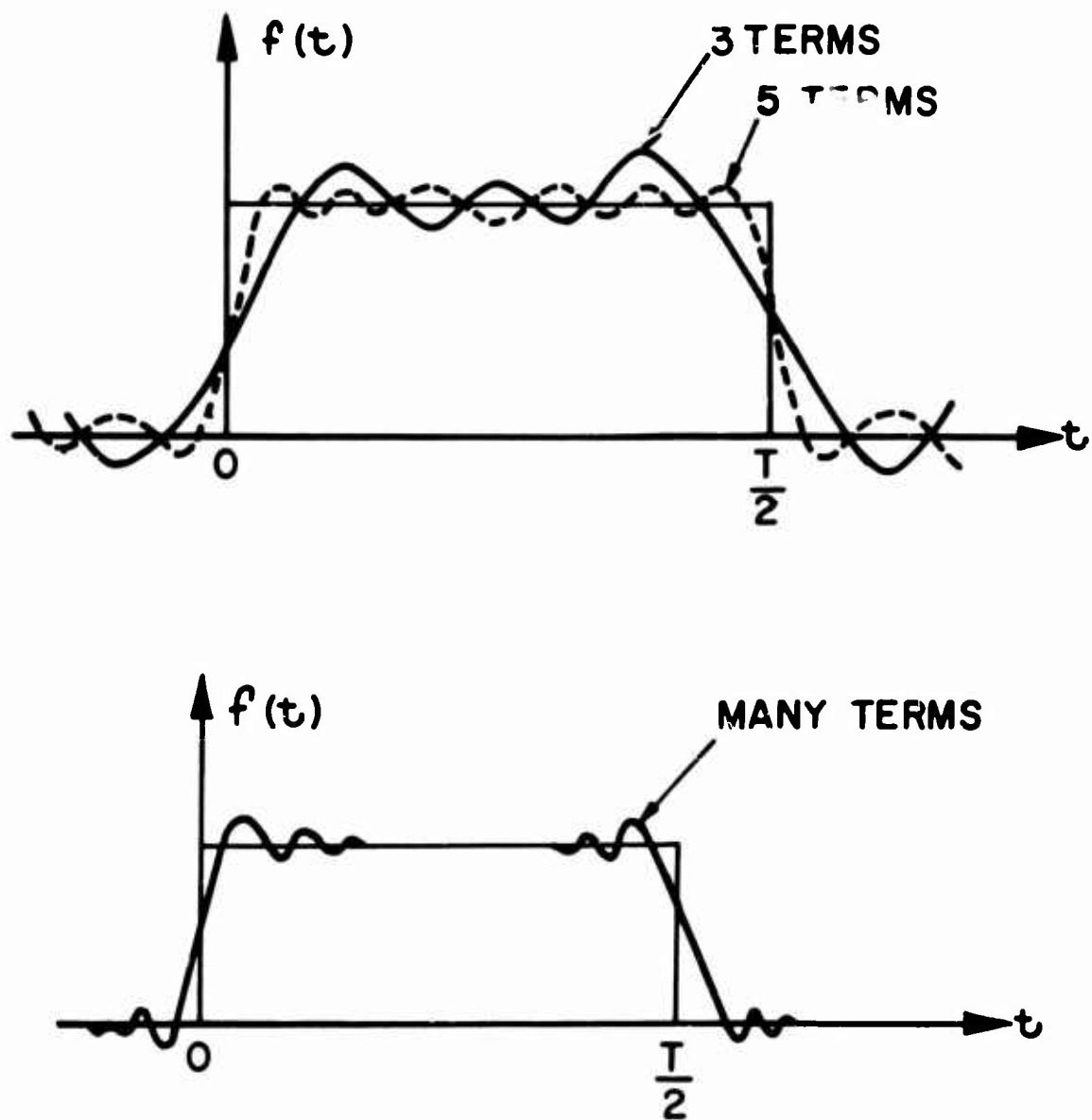
Figure 2. Fourier Series Approximations for Rectangular Wave.

SECTION 3

DISCUSSION OF ACCELERATION INPUTS

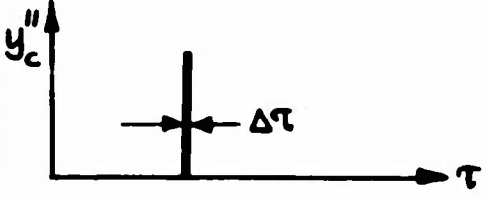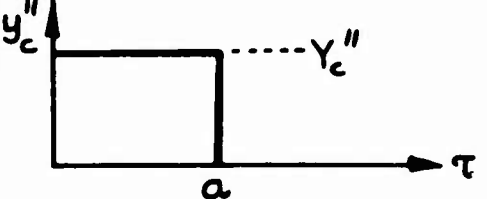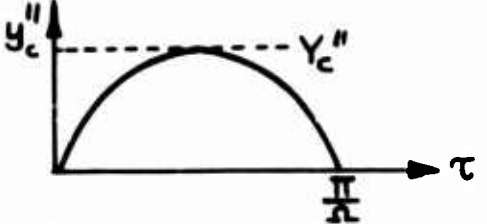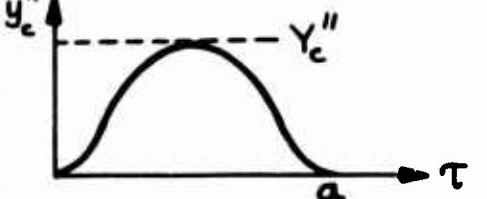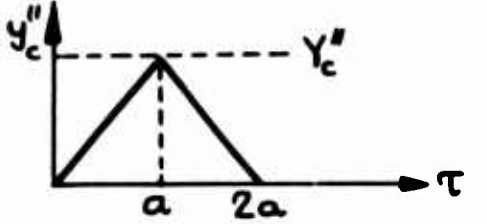Most idealized acceleration inputs fall into one of three classes:

(i) discrete pulses

(ii) oscillatory wave-forms

and (iii) semi-infinite ramps.

In this paper, the first class has been investigated in greatest detail; using the Laplace transform technique, the response of the basic system to several different types of pulse has been analyzed. For each pulse, a family of curves is presented, showing the variation of amplification factor with pulse duration, for a range of values of the damping coefficient ratio. These are plotted in Figures 4 through 8. Amplification factor is defined here as the ratio of maximum dynamic deflection to static deflection. The pulses considered, together with their Laplace transforms, are summarized in Table 1 of this section; the detailed solutions will be found in the Appendices.

Oscillatory inputs may also be solved using Laplace transforms; however, a more convenient approach in general is by way of the Fourier series expansion of the input function. This method is discussed in detail in Section 2 of this paper. The Fourier series representations of some periodic functions are presented in Table 2.

Semi-infinite ramps represent acceleration inputs which start from an initial value of zero, and rise in some fashion to a constant acceleration level. Rise-time is the most important parameter for this type of input. Table 3 shows some of the ramps commonly encountered, together with their Laplace transforms.

11

TABLE 1: DISCRETE PULSES.

TYPE	FUNCTION	L - TRANSFORM
DIRAC IMPULSE 	$y_c'' = (\Delta v)(\delta\tau)$	$\mathcal{L}(y_c'') = \Delta v$
RECTANGULAR PULSE 	$y_c'' = Y_c'' \quad (0 < \tau < a)$ $= 0 \quad (\tau > a)$	$\mathcal{L}(y_c'') = \dfrac{Y_c''}{p}\left(1 - e^{-ap}\right)$
HALF-SINE PULSE 	$y_c'' = Y_c'' \sin \Omega\tau$ $\left(0 < \tau < \dfrac{\pi}{\Omega}\right)$ $= 0 \quad \left(\tau > \dfrac{\pi}{\Omega}\right)$	$\mathcal{L}(y_c'')$ $= \dfrac{Y_c'' \, \Omega}{p^2 + \Omega^2}\left(1 + e^{-\frac{p\pi}{\Omega}}\right)$
VERSED-SINE PULSE 	$y_c'' = \dfrac{Y_c''}{2}\left(1 - \cos\dfrac{2\pi}{a}\tau\right)$ $(0 < \tau < a)$ $= 0 \quad (\tau > a)$	$\mathcal{L}(y_c'') =$ $\dfrac{Y_c''}{2}\dfrac{\left(\frac{2\pi}{a}\right)^2\left(1 - e^{-ap}\right)}{p\left(p^2 + \left[\frac{2\pi}{a}\right]^2\right)}$
TRIANGULAR PULSE 	$y_c'' = \dfrac{Y_c''}{a}\tau \quad (0 < \tau < a)$ $= \dfrac{Y_c''}{a}(2a - \tau) \quad (a < \tau < 2a)$ $= 0 \quad (\tau > 2a)$	$\mathcal{L}(y_c'') =$ $\dfrac{Y_c''}{ap^2}\left(1 - e^{-ap}\right)^2$

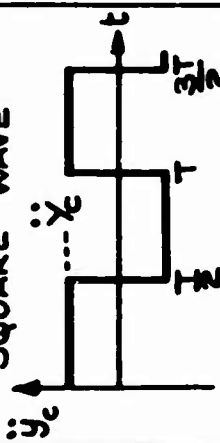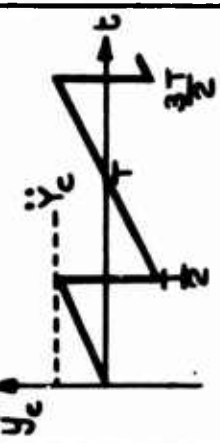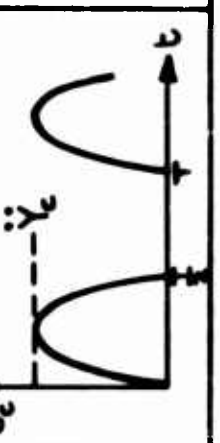# TABLE 2 : REPRESENTATION OF SOME PERIODIC FUNCTIONS BY FOURIER SERIES.
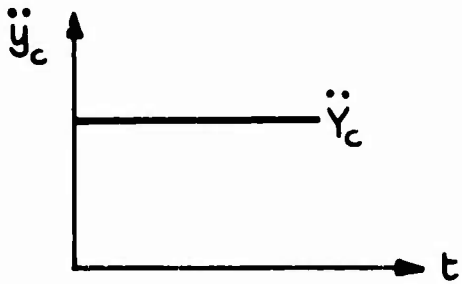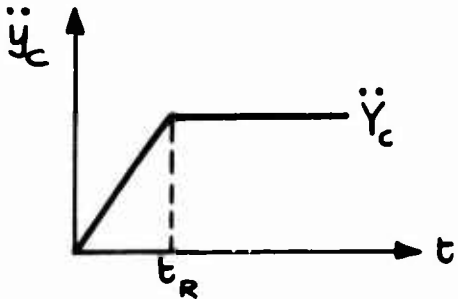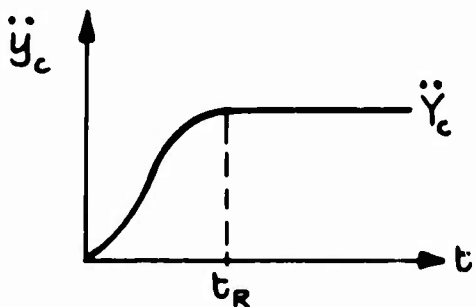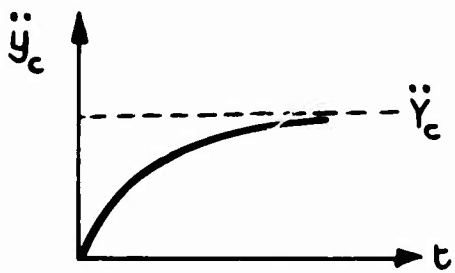
TYPE	FUNCTION	FOURIER SERIES
SQUARE WAVE	$\ddot{y}_c = \ddot{Y}_c,\ (0<t<\frac{T}{2})$ $= -\ddot{Y}_c,\ (\frac{T}{2}<t<T)$ $\ddot{y}_c(t) = \ddot{y}_c(t+T)$	$\ddot{y}_c = \dfrac{4\ddot{Y}_c}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)\Omega t}{2n-1}$
TRIANGULAR WAVE	$\ddot{y}_c = \dfrac{2\ddot{Y}_c}{T}t, (0<t<\frac{T}{2})$ $= -\dfrac{2\ddot{Y}_c}{T}(T-t), (\frac{T}{2}<t<T)$ $\ddot{y}_c(t) = \ddot{y}_c(t+T)$	$\ddot{y}_c = \dfrac{2\ddot{Y}_c}{\pi}\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\dfrac{\sin n\Omega t}{n}$
RECTIFIED SINE-WAVE	$\ddot{y}_c = \ddot{Y}_c\left|\sin\dfrac{2\pi}{T}t\right|$	$\ddot{y}_c = \dfrac{2\ddot{Y}_c}{\pi}\left\{1-2\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos 2n\Omega t}{4n^2-1}\right\}$
RECTIFIED HALF-SINE WAVE	$\ddot{y}_c = \ddot{Y}_c\sin\dfrac{2\pi}{T}t, (0<t<\frac{T}{2})$ $= 0,\ (\frac{T}{2}<t<T)$ $\ddot{y}_c(t) = \ddot{y}_c(t+T)$	$\ddot{y}_c = \dfrac{\ddot{Y}_c}{2}\sin\Omega t + \dfrac{\ddot{Y}_c}{\pi}\left\{1-2\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos 2n\Omega t}{4n^2-1}\right\}$

13

TABLE 3: SEMI-INFINITE RAMPS.

TYPE	FUNCTION	L-TRANSFORM
VERTICAL FRONT 	$\ddot{y}_c = \ddot{Y}_c \quad (t>0)$	$\mathcal{L}(\ddot{y}_c) = \dfrac{\ddot{Y}_c}{P}$
LINEAR FRONT 	$\ddot{y}_c = \dfrac{\ddot{Y}_c}{t_R} t \quad (0<t<t_R)$ $= \ddot{Y}_c \quad (t>t_R)$	$\mathcal{L}(\ddot{y}_c) =$ $\dfrac{\ddot{Y}_c}{t_R p^2}\left(1-e^{-pt}\right)$
VERSED-SINE FRONT 	$\ddot{y}_c = \dfrac{\ddot{Y}_c}{2}\left(1-\cos\dfrac{\pi t}{t_R}\right)$ $(0<t<t_R)$ $= \ddot{Y}_c \quad (t>t_R)$	$\mathcal{L}(\ddot{y}_c) =$ $\dfrac{\ddot{Y}_c \left[\frac{\pi}{t_R}\right]^2\left(1+e^{-t_R p}\right)}{2p\left(p^2+\left[\frac{\pi}{t_R}\right]^2\right)}$
EXPONENTIAL FRONT 	$\ddot{y}_c = \ddot{Y}_c\left(1-e^{-at}\right)$ $(t>0)$	$\mathcal{L}(\ddot{y}_c) = \dfrac{\ddot{Y}_c \, a}{P(p+a)}$

SECTION 4

REFERENCES

1.   Mindlin, R. D., F. W. Stubner and H. L. Cooper, "Response of Damped Elastic Systems to Transient Disturbances," Proceedings of the Society for Experimental Stress Analysis, Vol. 5, No. 2, 1948.

2.   Ayre, R. S., "Transient Response to Step and Pulse Functions," Shock and Vibration Handbook, Vol. I, Chap. 8, McGraw-Hill Book Co., Inc., New York, N.Y., 1961.

3.   Morse, P. M. and H. Feshbach, Method of Theoretical Physics, Part 1, Chap. 6, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

4.   Payne, P. R. and E. L. Stech. Human Body Dynamics under Short-Term Acceleration; Frost Engineering Development Corporation Technical Report No. 115-2, June 1962.

APPENDIX I

DIRAC IMPULSE FUNCTION

From Section 3, the equation for the Dirac impulse function is simply

$$y_c'' = \Delta v \, \Delta \tau \tag{35}$$

with the Laplace transform

$$\mathcal{L}(y_c'') = \Delta v \tag{36}$$

In (19), (replacing y_c'' by $\Delta v/\omega$)

$$\frac{\omega^2 \delta}{\omega \Delta v} = \mathcal{L}^{-1}\left[\frac{1}{(p+\bar{c})^2 + \eta^2}\right] \tag{37}$$

$$\therefore \frac{\omega^2 \delta}{\omega \Delta v} = \frac{e^{-\bar{c}\tau}}{\eta} \sin \eta \tau \tag{38}$$

Equation (38) is plotted in Figure 3 for several values of $\bar{c}$. It is interesting to note that in the initial phases of the motion it is adequately described by a system consisting of a mass and damper:

i.e.

$$f(\delta) = \frac{1}{2\bar{c}}\left(1 - e^{-2\bar{c}\tau}\right) \tag{39}$$

As the deflection increases, the spring force becomes important enough to cause a divergence from the simple damper result, and returns the system to equilibrium, after passing through a maximum deflection.

We determine the maximum deflection by differentiating eq. (38) with respect to τ, then equating to zero.

i.e.

$$\frac{d}{d\tau}\left\{\frac{\omega^2 \delta}{\omega \Delta v}\right\} = e^{-\bar{c}\tau} \cos \eta \tau - \frac{\bar{c}}{\eta} e^{-\bar{c}\tau} \sin \eta \tau = 0$$

$$\therefore 1 - \frac{\bar{c}}{\eta} \tan \eta \tau^* = 0 \tag{40}$$

Solving for τ^*, the time at which maximum deflection occurs, gives

$$\tau^* = \frac{1}{\eta} \arctan \frac{\eta}{-\bar{c}} = \frac{\arcsin \eta}{\eta} = \frac{\varphi}{\eta}$$

(Note that $\tau^* \longrightarrow 1$ as $\bar{c} \longrightarrow 1$

$\tau^* \longrightarrow \frac{\pi}{2}$ as $\bar{c} \longrightarrow 0$)

Substituting for τ^{*} in eq. (38),

$$\frac{\omega^{2}\delta_{max}}{\omega\,\Delta v} = \frac{e^{-\frac{\bar{c}\,\varphi}{\eta}}}{\eta}\,\sin\left(\eta\;\frac{arcsin\,\eta}{\eta}\right)$$

$$= e^{-\bar{c}\varphi/\eta} \tag{41}$$

Note that $\quad\dfrac{\omega^{2}\delta_{max}}{\omega\,\Delta v} \longrightarrow \dfrac{1}{e}\;$ as $\;\bar{c}\longrightarrow 1$

$$\longrightarrow 1\;\text{ as }\;\bar{c}\longrightarrow 0$$

τ^{*} and $\dfrac{\omega^{2}\delta_{max}}{\omega\,\Delta v}$ are plotted in Fig. 4.

When the initial deflection is not zero, eq. (38) may be written

$$\frac{\omega^{2}\delta}{\omega\,\Delta v} = e^{-\bar{c}\tau}\left\{\xi\cos\eta\tau +\left[\frac{\bar{c}}{\eta}\xi + \frac{1}{\eta}\right]\sin\eta\tau\right\} \tag{42}$$

using the substitution $\xi = \dfrac{\omega^{2}\delta_{o}}{\omega\,\Delta v}$ for brevity,

where $\delta_{o} =$ initial spring deflection.

This can be simplified to

$$\left.\begin{array}{c}\dfrac{\omega^{2}\delta}{\omega\,\Delta v} = \dfrac{e^{-\bar{c}\tau}}{\eta}\sqrt{\xi^{2}+2\bar{c}\xi+1}\;\sin(\eta\tau+\theta)\\[2em] \text{where } \theta = arcsin\dfrac{\xi\eta}{\sqrt{\xi^{2}+2\bar{c}\xi+1}}\end{array}\right\} \tag{43}$$

Differentiating with respect to τ,

$$\frac{d}{d\tau}\left\{\frac{\omega^{2}\delta}{\omega\,\Delta v}\right\} = \frac{\sqrt{\xi^{2}+2\bar{c}\xi+1}}{\eta}\left\{\eta e^{-\bar{c}\tau}\cos(\eta\tau+\theta)\right.$$

$$\left.-\bar{c}e^{-\bar{c}\tau}\sin(\eta\tau+\theta)\right\}$$

$$= \frac{e^{-\bar{c}\tau}}{\eta}\sqrt{\xi^{2}+2\bar{c}\xi+1}\;\sin(\eta\tau+\theta+\varphi) \tag{44}$$

where $\sin\varphi = \eta$ as before.

18

Equating to zero, this simplifies to

$$\eta \tau^* + \Theta + \eta = n\pi \tag{45}$$

$$\therefore \sin(\eta \tau^* + \Theta) = \sin(n\pi - \varphi)$$

$$= \pm \eta \tag{46}$$

Also

$$\tau^* = \frac{n\pi - \Theta - \varphi}{\eta} \tag{47}$$

Substituting eq. (46) and (47) into eq. (43) gives

$$\left. \begin{array}{l}
\dfrac{\omega^2 \delta_{max}}{\omega \, \Delta v} = e^{-\frac{\xi}{\eta}(\varphi - \Theta)} \sqrt{\xi^2 + 2\bar{c}\,\xi + 1} \\[4mm]
\text{where} \quad \sin \Theta = \dfrac{\xi \eta}{\sqrt{\xi^2 + 2\bar{c}\,\xi + 1}} \quad \left(0 \leqslant \Theta \leqslant \frac{\pi}{2}\right) \\[4mm]
\quad \qquad \sin \varphi = \eta \qquad\qquad\qquad \left(\frac{\pi}{2} \leqslant \varphi \leqslant \frac{3\pi}{2}\right)
\end{array} \right\} \tag{48}$$

The response of an undamped system to an impulsive velocity change, for various values of the initial parameter ξ, is given in Figure 5.
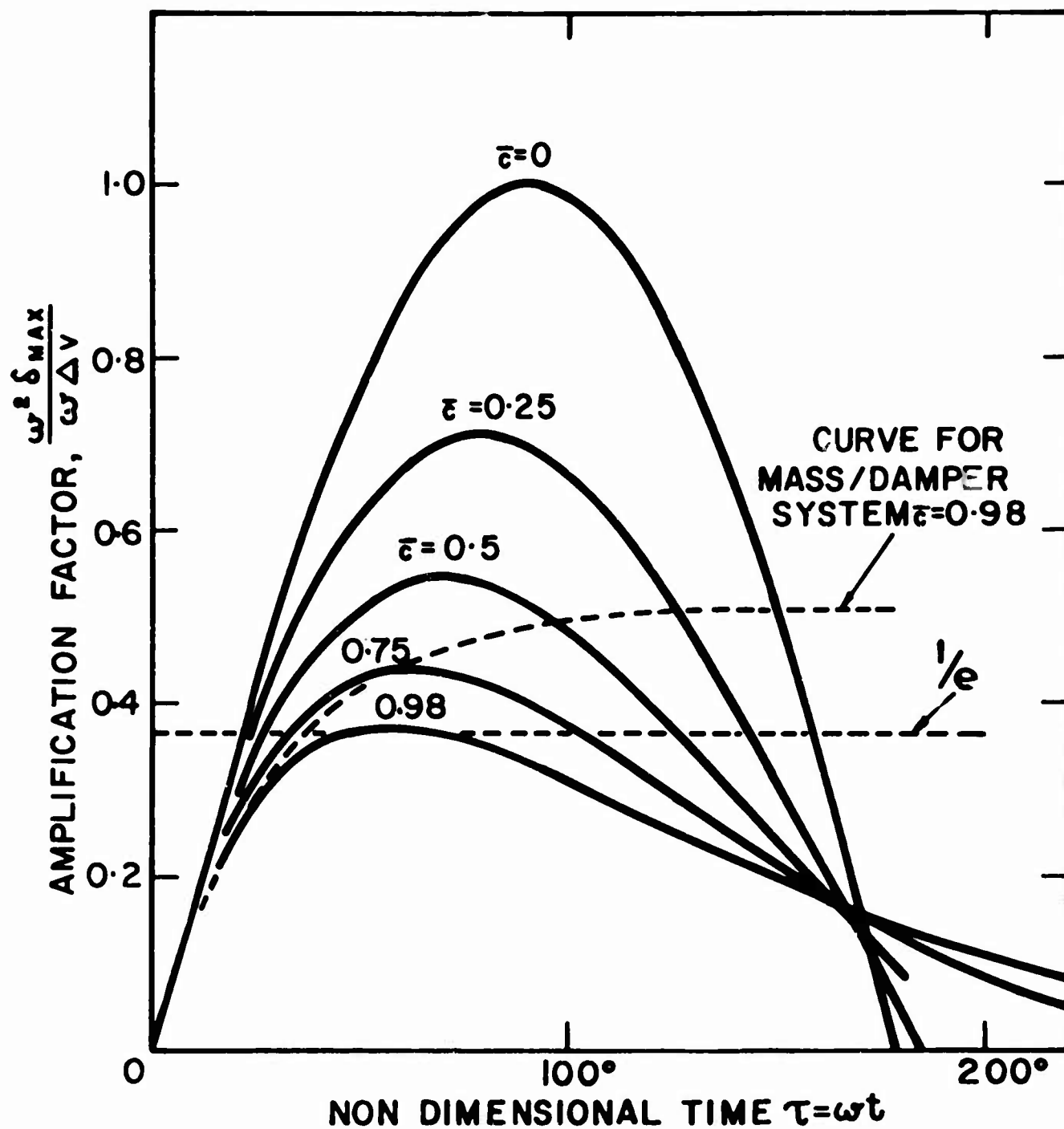
Figure 3. Response of a Damped Single Degree of
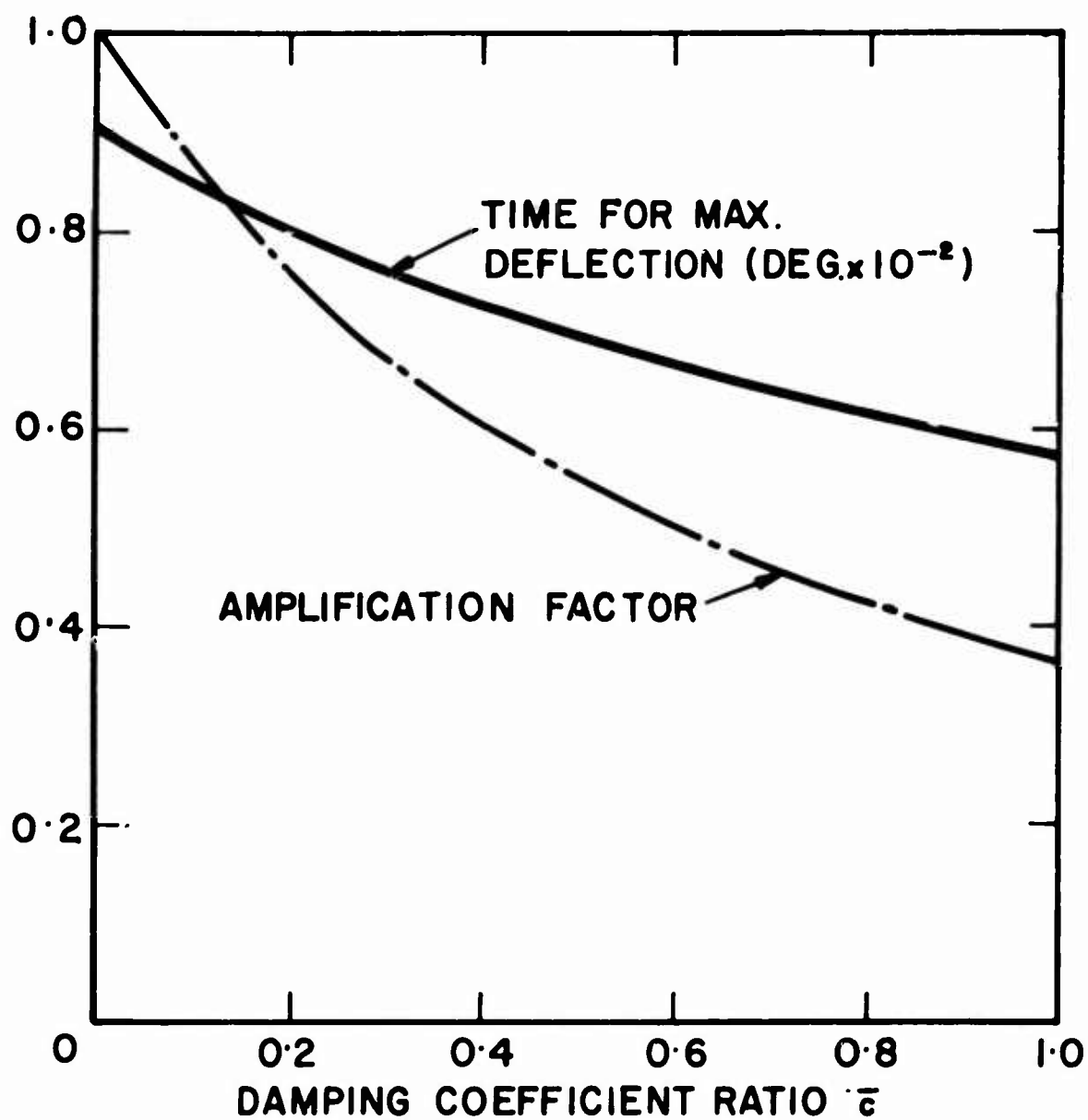Freedom System to an Impulsive Velocity Change.

Figure 4. Variation of Amplification Factor and Time of Maximum
Deflection with Damping Coefficient Ratio, for an Impulsively
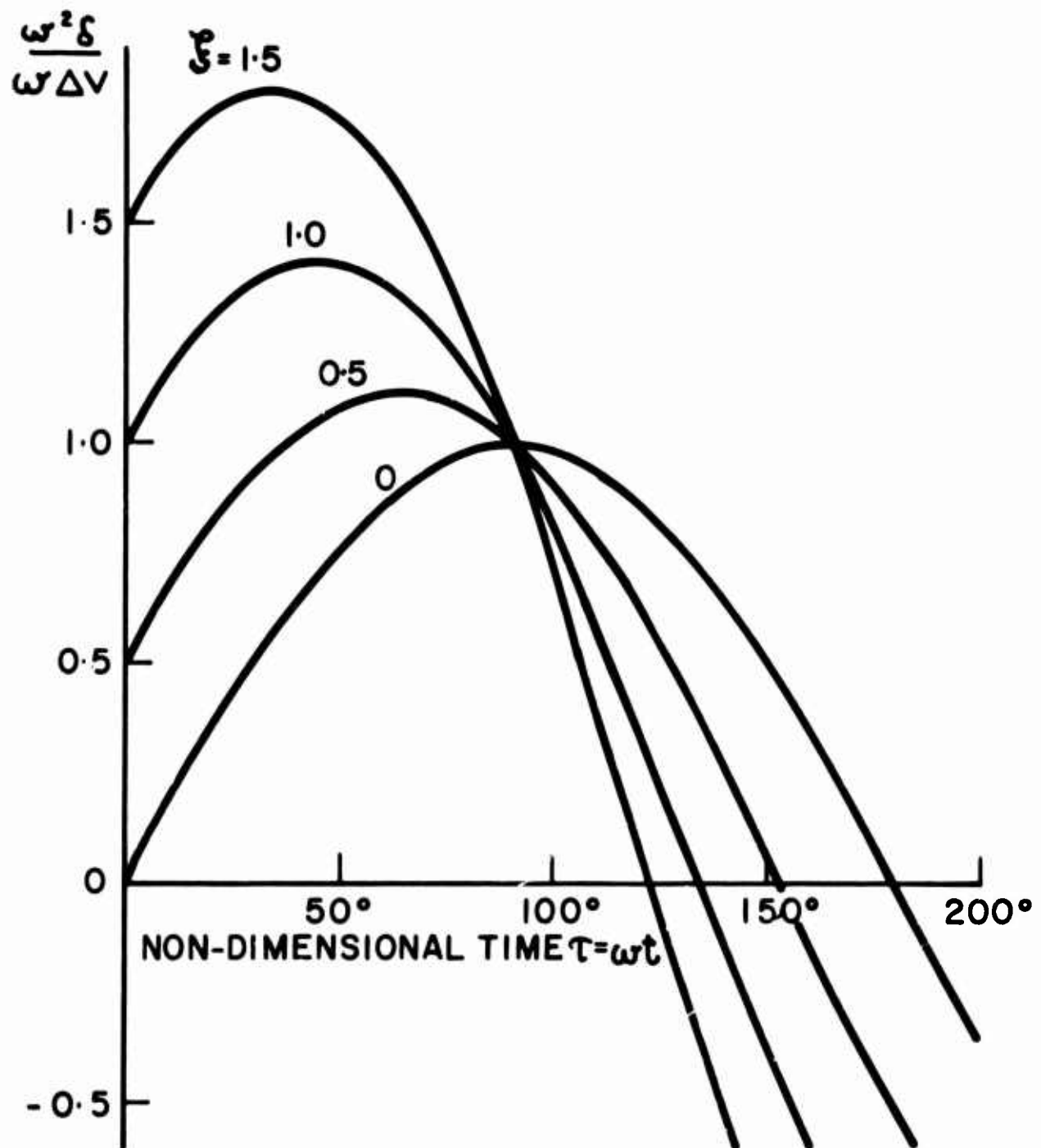Excited Linear System.

Figure 5. Influence of Initial Deflection Parameter
on Response of the System to an Impulse.

RECTANGULAR PULSE

From Section 3,

$$y_c'' = Y_c'' \qquad (0 \leqslant \tau < a)$$
$$= 0 \qquad (\tau > a) \qquad (49)$$

$$\mathcal{L}(y_c'') = \frac{Y_c''}{p}\left(1 - e^{-ap}\right) \qquad (50)$$

In eq. (19),

$$\frac{\delta}{Y_c''} = \mathcal{L}^{-1} \frac{1 - e^{-ap}}{p(p^2 + 2\bar{c}p + 1)} \qquad (51)$$

$$= 1 + \frac{e^{-\bar{c}\tau}}{\eta} \sin\left(\eta\tau - \arctan\frac{\eta}{-\bar{c}}\right)$$

$$-\left\{1 + \frac{e^{-\bar{c}(\tau-a)}}{\eta}\sin\left(\eta[\tau-a] - \arctan\frac{\eta}{-\bar{c}}\right)\right\} H(\tau-a)$$

where $H(\tau-a)$ denotes the Heaviside unit function:

$$H(\tau-a) = 0, \quad \tau < a$$
$$= 1, \quad \tau > a$$

$$\therefore \frac{\delta}{Y_c''} = 1 + \frac{e^{-\bar{c}\tau}}{\eta} \sin\left(\eta\tau - \arctan\frac{\eta}{-\bar{c}}\right) \qquad \text{for } \tau < a \quad (52)$$

and

$$\frac{\delta}{Y_c''} = \frac{1}{\eta}\left\{ e^{-\bar{c}\tau} \sin\left(\eta\tau - \arctan\frac{\eta}{-\bar{c}}\right) \right.$$

$$\left. -e^{-\bar{c}(\tau-a)}\sin\left(\eta[\tau-a] - \arctan\frac{\eta}{-\bar{c}}\right) \right\} \quad \text{for } \tau > a \quad (53)$$

Differentiating,

$$\frac{d}{d\tau}\left(\frac{\delta}{Y_c''}\right) = \frac{\delta'}{Y_c''} = -\frac{\bar{c}}{\eta} e^{-\bar{c}\tau} \sin\left(\eta\tau - \arctan\frac{\eta}{-\bar{c}}\right)$$

$$+ e^{-\bar{c}\tau} \cos\left(\eta\tau - \arctan\frac{\eta}{-\bar{c}}\right)$$

$$= \frac{e^{-\bar{c}\tau}}{\eta} \sin\eta\tau \qquad (\tau < a) \qquad (54)$$

and for the second time-regime,

$$\frac{\delta'}{Y_c''} = \frac{e^{-\bar{c}\tau}}{\eta} \sin \eta\tau - \frac{e^{-\bar{c}(\tau-a)}}{\eta} \sin \eta(\tau-a) \qquad (\tau > a) \quad (55)$$

We can find τ^*, the time at which δ_{max} occurs by setting eq. (55) equal to zero, and solving for time. The value of δ_{max} is then found by substituting $\tau = \tau^*$ into the equations for δ.

(i) <u>For $\tau < a$</u>, setting $\delta'/Y_c'' = 0$ gives

$$\sin \eta\tau^* = 0$$
$$\eta\tau^* = n\pi$$
$$\tau^* = \frac{n\pi}{\eta} \qquad (56)$$

In (53),
$$\frac{\delta_{max}}{Y_c''} = 1 + \frac{e^{-\frac{\bar{c}}{\eta}n\pi}}{\eta} \sin\left(n\pi - \arctan\frac{\eta}{-\bar{c}}\right)$$

and for $n = 1$, we have

$$\frac{\delta_{max}}{Y_c''} = 1 + e^{-\frac{\bar{c}\pi}{\eta}} \qquad (\tau < a) \qquad (57)$$

(Note that for zero damping, we get the familiar result $\delta_{max}/Y_c'' = 2$, indicating 100% overshoot of the system.)

(ii) <u>For $\tau > a$</u>, eq. (55) leads to

$$e^{-\bar{c}\tau^*}\left\{ \sin \eta\tau^* - e^{\bar{c}a} \sin \eta(\tau^* - a) \right\} = 0$$

and, ignoring the trivial result of $\tau^* = \infty$, we get

$$\sin \eta\tau^* = e^{\bar{c}a}\left\{ \sin \eta\tau^* \cos \eta a - \cos \eta\tau^* \sin \eta a \right\}$$

Dividing through by $\sin \eta\tau^*$,

$$1 = e^{\bar{c}a}\left\{ \cos \eta a - \cot \eta\tau^* \sin \eta a \right\}$$

$$\therefore \cot \eta\tau^* = \frac{\cos \eta a - e^{-\bar{c}a}}{\sin \eta a}$$

whence
$$\tau^* = \frac{1}{\eta} \arctan\left\{ \frac{\sin \eta a}{\cos \eta a - e^{-\bar{c}a}} \right\} \qquad (58)$$

In (53),

$$\frac{\delta_{max}}{y_c{''}} = e^{-\bar{c}\tau^*} \sin\left(\eta\tau^* - \arctan\frac{\eta}{-\bar{c}}\right)$$
$$- e^{-\bar{c}(\tau^* - a)} \sin\left(\eta[\tau^* - a] - \arctan\frac{\eta}{-\bar{c}}\right) \qquad (59)$$
$$(\tau > a)$$

The response of a damped single degree of freedom system to a rectangular-pulse acceleration input is plotted in Figure 6, as $\omega^2\delta_{max}/y_c{''}$ versus the non-dimensional pulse duration, ωa

Using Figure 4, it is possible to calculate the initial slope of the plot of $\frac{\omega^2\delta_{max}}{y_c{''}}$ vs. ωa for any arbitrary pulse, providing a useful check on the solution; this check was applied to the case of the rectangular pulse.
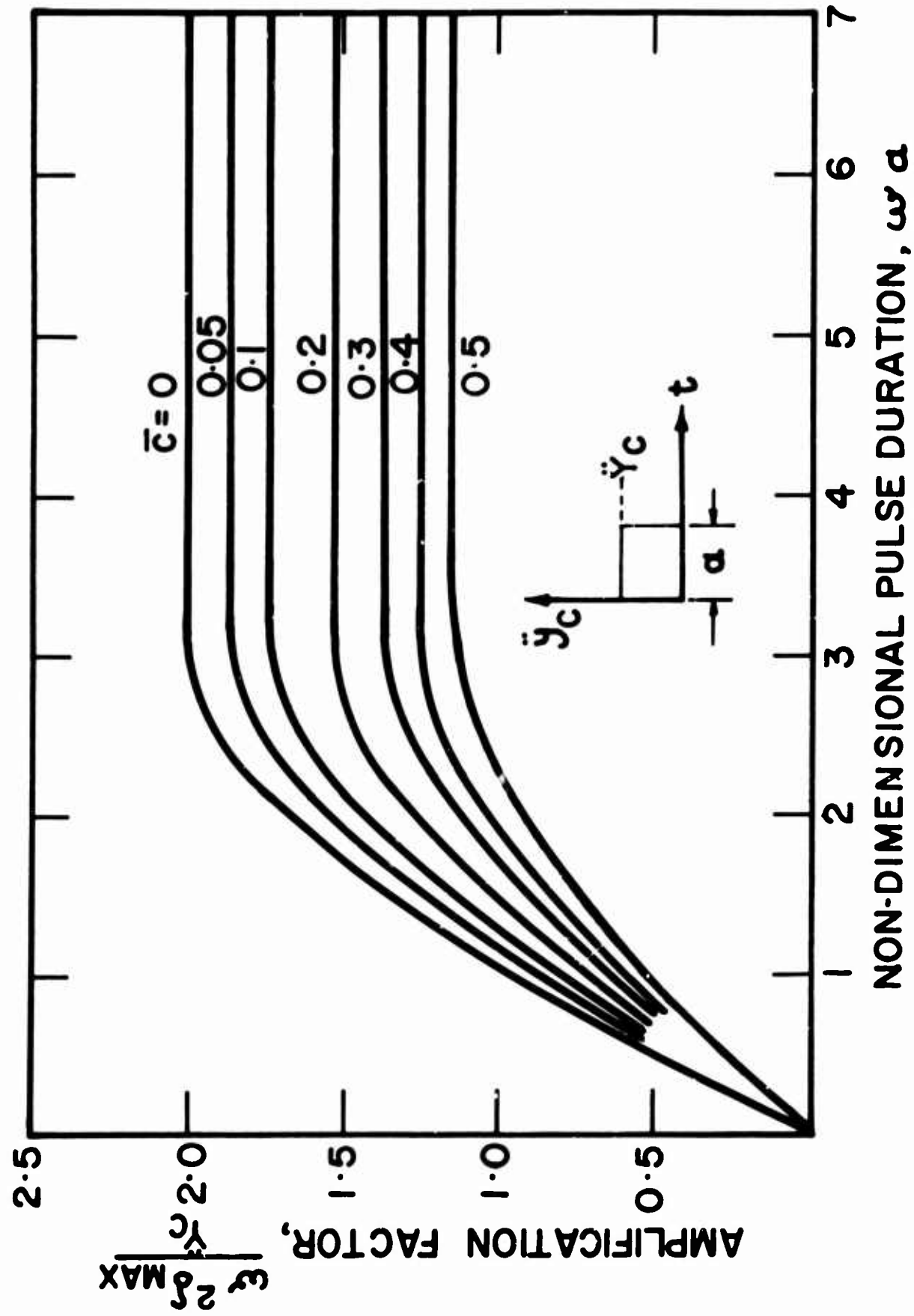
Figure 6. Response of the System to a Rectangular Pulse (Zero Initial Conditions).

APPENDIX III

HALF-SINE PULSE

The equation of the pulse, from Section 3, is

$$y_c'' = Y_c'' \sin \Omega\tau \qquad (0 \leqslant \tau \leqslant \tfrac{\pi}{\Omega})$$
$$= 0 \qquad (\tau > \tfrac{\pi}{\Omega}) \qquad (60)$$

with the Laplace transform

$$\mathcal{L}(y_c'') = Y_c'' \frac{\Omega(1 + e^{-P\frac{\pi}{\Omega}})}{P^2 + \Omega^2} \qquad (61)$$

Substituting eq. (61) into eq. (19) gives

$$\frac{\delta}{Y_c''} = \mathcal{L}^{-1}\left\{\frac{\Omega(1 + e^{-P\frac{\pi}{\Omega}})}{(P^2 + \Omega^2)(P^2 + 2\bar{z}P + 1)}\right\} \qquad (62)$$

In terms of the Heaviside unit function,

$$\frac{\delta}{Y_c''} = \frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{z}^2\Omega^2}}\left\{\frac{1}{\Omega} \sin\left(\Omega\tau - \gamma_1\right)\right.$$
$$+ \frac{e^{-\bar{z}\tau}}{\eta} \sin(\eta\tau - \gamma_2) + \frac{1}{\Omega} \sin\left(\Omega\left[\tau - \tfrac{\pi}{\Omega}\right] - \gamma_1\right)H\left(\tau - \tfrac{\pi}{\Omega}\right)$$
$$+ \left. \frac{e^{-\bar{z}(\tau - \frac{\pi}{\Omega})}}{\eta} \sin\left(\eta\left[\tau - \tfrac{\pi}{\Omega}\right] - \gamma_2\right) H\left(\tau - \tfrac{\pi}{\Omega}\right)\right\}(63)$$

The phase angles are

$$\gamma_1 = \arctan \frac{2\bar{z}\Omega}{1 - \Omega^2} \qquad (0 \leqslant \gamma_1 \leqslant \pi)$$
$$\gamma_2 = \arctan \frac{-2\bar{z}\eta}{\Omega^2 - 1 + 2\bar{z}^2} \qquad (\pi \leqslant \gamma_2 \leqslant 2\pi)^{(64)}$$

We must consider separate solutions for the two time regimes

$$(i)\ \tau < \tfrac{\pi}{\Omega}, \qquad (ii)\ \tau > \tfrac{\pi}{\Omega}$$

(i) $\underline{\tau < \pi/\Omega}$

Because of the Heaviside unit function, eq. (63) becomes

$$\frac{\delta}{y_c''} = \frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{c}^2\Omega^2}} \left\{ \frac{1}{\Omega} \sin(\Omega\tau - \psi_1) + \frac{e^{-\bar{c}\tau}}{\eta} \sin(\eta\tau - \psi_2) \right\} \quad (65)$$

Note that for $\bar{c} = 0$, this simplifies to

$$\frac{\delta}{y_c''} = \frac{1}{1 - \Omega^2} \left\{ \sin \Omega\tau - \Omega \sin \tau \right\} \quad (66)$$

since $\psi_1 \to 0$ and $\psi_2 \to \pi$ as $\bar{c} \to 0$. This agrees with the zero-damping solution for $\tau < \pi/\Omega$ established in, for example, Ref. 4. Differentiating (66),

$$\frac{\delta'}{y_c''} = \frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{c}^2\Omega^2}} \left\{ \cos(\Omega\tau - \psi_1) + \frac{e^{-\bar{c}\tau}}{\eta} \sin(\eta\tau - \psi_2 + \psi_3) \right\}$$

Where $\psi_3 = \arctan\left(\frac{\eta}{\bar{c}}\right) = \arcsin \eta \qquad (\pi/2 \leqslant \psi_3 < \pi) \quad (67)$

τ^*, the time at which δ achieves its maximum value, is the solution of the equation obtained by setting (67) to zero:

$$\frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{c}^2\Omega^2}} \left\{ \cos(\Omega\tau^* - \psi_1) + \frac{e^{-\bar{c}\tau^*}}{\eta} \sin(\eta\tau^* - \psi_2 + \psi_3) \right\} = 0 \quad (68)$$

An explicit solution of eq. (68) in τ^*, if it exists, is certainly not simple to find, except for the case of $\bar{c} = 0$, when eq. (68) becomes

$$\cos \Omega\tau^* - \cos \tau^* = 0$$

so that

$$\cos \Omega\tau^* = \cos \tau^*$$

$$\Omega\tau^* = 2\pi n \pm \tau^*$$

whence, for $n = 1$, $\qquad \tau^* = \dfrac{2\pi}{\Omega \pm 1} \quad (69)$

Since τ^* is necessarily positive, and Ω may be less than unity, we must take the positive sign in the denominator of eq. (69).

For non-zero values of $\bar{c}$, τ^* is found most directly by a numerical-graphical approach.

(ii) $\underline{\tau > \pi/\Omega}$

For this case, since $(\tau - \pi/\Omega) > 0$, the value of the Heaviside unit function is unity, and eq. (68) becomes

$$\frac{\delta}{Y_c{''}} = \frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{c}^2\Omega^2}} \left\{ \frac{1}{\Omega} \sin(\Omega\tau - \psi_1) \right.$$

$$+ \frac{e^{-\bar{c}\tau}}{\eta} \sin(\eta\tau - \psi_2)$$

$$+ \frac{1}{\Omega} \sin\left(\Omega\left[\tau - \frac{\pi}{\Omega}\right] - \psi_1\right)$$

$$\left. + \frac{e^{-\bar{c}(\tau - \pi/\Omega)}}{\eta} \sin\left(\eta\left[\tau - \frac{\pi}{\Omega}\right] - \psi_2\right)\right\} \quad (70)$$

Noting that
$$\sin\left(\Omega\left[\tau - \frac{\pi}{\Omega}\right] - \psi_1\right) = \sin(\Omega\tau - \psi_1 - \pi)$$
$$= -\sin(\Omega\tau - \psi_1) ,$$

we may simplify eq. (70) to

$$\frac{\delta}{Y_c{''}} = \frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{c}^2\Omega^2}} \frac{e^{-\bar{c}\tau}}{\eta} \left\{ \begin{array}{l} \sin(\eta\tau - \psi_2) \\ + e^{\bar{c}\pi/\Omega} \sin\left(\eta\tau - \psi_2 - \eta\frac{\pi}{\Omega}\right) \end{array}\right\} \quad (71)$$

Differentiating eq. (71) with respect to τ gives

$$\frac{\delta'}{Y_c{''}} = \frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{c}^2\Omega^2}} \frac{e^{-\bar{c}\tau}}{\eta} \left\{ -\bar{c} \sin(\eta\tau - \psi_2) \right.$$

$$- \bar{c} e^{\bar{c}\pi/\Omega} \sin\left(\eta\tau - \psi_2 - \eta\pi/\Omega\right) + \eta \cos(\eta\tau - \psi_2)$$

$$\left. + \eta e^{\bar{c}\pi/\Omega} \sin\left(\eta\tau - \psi_2 - \eta\pi/\Omega\right)\right\} \quad (72)$$

Now
$$\eta \cos X - \bar{c} \sin X = \sqrt{\eta^2 + \bar{c}^2} \sin\left(X + \arctan\frac{\eta}{\bar{c}}\right)$$
$$= \sin(X + \psi_3)$$

by the definition of ψ_3 in eq. (67).

$\therefore$ (72) becomes

$$\frac{\delta'}{Y_c{''}} = \frac{\Omega}{\sqrt{(1-\Omega^2)^2 + 4\bar{c}^2\Omega^2}} \frac{e^{-\bar{c}\tau}}{\eta} \left\{ \sin(\eta\tau - \psi_2 + \psi_3) \right.$$

$$\left. + e^{\bar{c}\pi/\Omega} \sin\left(\eta\tau - \psi_2 + \psi_3 - \eta\pi/\Omega\right)\right\} \quad (73)$$

29

Equating to zero, we have the equation for τ^*:

$$\sin\left(\eta\tau^* - \gamma_2 + \gamma_3\right) + e^{\bar{c}\pi/\Omega}\sin\left(\eta\tau^* - \gamma_2 + \gamma_3 - \eta\pi/\Omega\right) = 0, \quad (74)$$

ignoring the trivial solution of $e^{-\bar{c}\tau^*} = 0$.

       We can again make the useful check of inspecting the zero damping case: for $\bar{c} = 0$, eq. (71) becomes

$$\frac{\delta}{\gamma_c''} = \frac{\Omega}{1-\Omega^2}\left\{\sin(\tau-\pi) + \sin\left(\tau-\pi-\pi/\Omega\right)\right\}$$

$$= -\frac{\Omega}{1-\Omega^2}\left\{-\sin\tau - \sin\left(\tau-\pi/\Omega\right)\right\} \quad (75)$$

The solution for this case given in Ref. 4 may easily be rearranged to give eq. (75).

Setting $\bar{c} = 0$ in eq. (73) gives

$$\sin\left(\tau^* - \pi + \pi/2\right) + \sin\left(\tau^* - \pi + \pi/2 - \pi/\Omega\right) = 0$$

or
$$\cos\tau^* + \cos\left(\tau^* - \pi/\Omega\right) = 0$$

$$\therefore \tau^* = n\pi \pm \left(\tau^* - \pi/\Omega\right)$$

$$2\tau^* = n\pi \pm \pi/\Omega$$

Finally, for $n = 1$,

$$\tau^* = \frac{\pi}{2} + \frac{1}{2}\frac{\pi}{\Omega} \quad (76)$$

Substituting for τ^* into eq. (72) leads to

$$\frac{\delta_{max}}{\gamma_c''} = \frac{-\Omega}{1-\Omega^2}\left\{\cos\frac{\pi}{2\Omega} + \cos\left(\frac{-\pi}{2\Omega}\right)\right\}$$

$$= \frac{-2\Omega}{1-\Omega^2}\cos\frac{\pi}{2\Omega} \quad (77)$$

which agrees with the solution for δ_{max}/γ_c'' given in Ref. 4.

      We need to solve eq. (74) explicitly for τ^*.

For brevity, write

$$A = \gamma_2 - \gamma_3$$
$$B = \gamma_2 - \gamma_3 + \eta \pi / \Omega$$
$$(78)$$

so that eq. (74) becomes

$$\sin\left(\eta \tau^* - A\right) + e^{\varepsilon \pi / \Omega} \sin\left(\eta \tau^* - B\right) = 0$$

$$\sin \eta \tau^* \cos A - \cos \eta \tau^* \sin A$$
$$+ e^{\varepsilon \pi / \Omega} \left\{ \sin \eta \tau^* \cos B - \cos \eta \tau^* \sin B \right\} = 0$$

$$\therefore \quad \sin \eta \tau^* \left\{ \cos A + e^{\varepsilon \pi / \Omega} \cos B \right\}$$
$$= \cos \eta \tau^* \left\{ \sin A + e^{\varepsilon \pi / \Omega} \sin B \right\}$$

$$\therefore \quad \tan \eta \tau^* = \frac{\sin A + e^{\varepsilon \pi / \Omega} \sin B}{\cos A + e^{\varepsilon \pi / \Omega} \cos B}$$

whence

$$\tau^* = \frac{1}{\eta} \arctan \left\{ \frac{\sin A + e^{\varepsilon \pi / \Omega} \sin B}{\cos B + e^{\delta \pi / \Omega} \cos B} \right\}$$

or, substituting for A and B,

$$\tau^* = \frac{1}{\eta} \arctan \frac{\sin(\gamma_2 - \gamma_3) + e^{\varepsilon \pi / \Omega} \sin(\gamma_2 - \gamma_3 + \eta \pi / \Omega)}{\cos(\gamma_2 - \gamma_3) + e^{\varepsilon \pi / \Omega} \cos(\gamma_2 - \gamma_3 + \eta \pi / \Omega)} \qquad (79)$$

Maximum deflection may now be derived by substituting for $\tau = \tau^*$ in eq. (71). The response of the system to a half-cycle sine pulse is plotted in Fig. 7, for a range of values of damping coefficient.
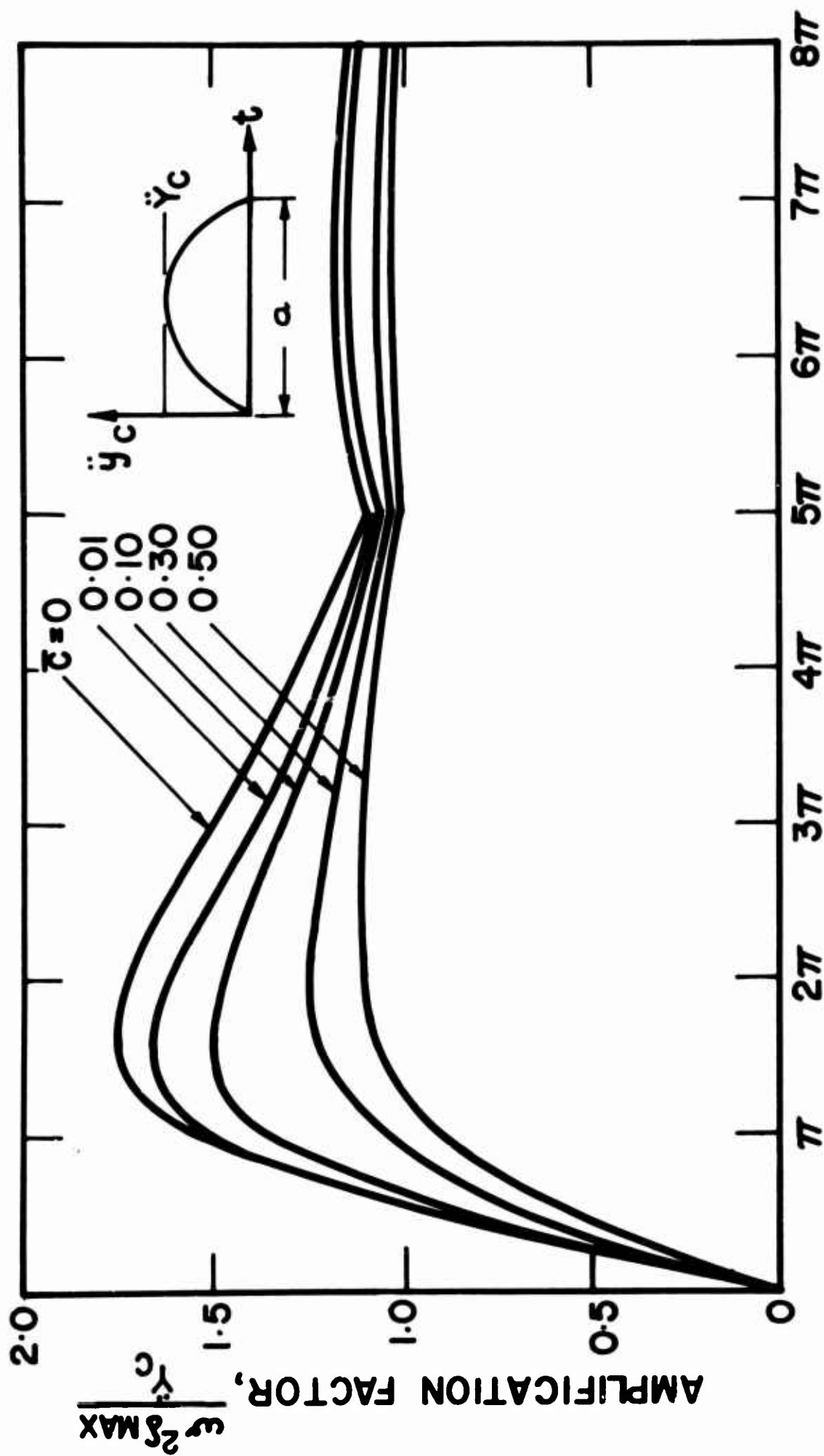
31

Figure 7. Response of the System to a Half-sine Pulse (Zero Initial Conditions).

VERSED-SINE PULSE

From Section 3,

$$y_c'' = \frac{y_c''}{2}\left[1 - \cos\frac{2\pi\tau}{a}\right]$$

$$\left.\begin{array}{l} (0 < \tau \leq a) \\ \\ = 0 \qquad (\tau > a) \end{array}\right\} \quad (80)$$

$$\mathcal{L}(y_c'') = \frac{y_c''}{2}\left[\frac{1}{p} - \frac{p}{p^2+\left(\frac{2\pi}{a}\right)^2}\right]\left(1-e^{-ap}\right) \quad (81)$$

Using eq. (19), and writing $\frac{2\pi}{a}$ as D, for convenience,

$$\frac{\delta}{y_c''} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1-e^{-ap}}{p[p^2+2\bar{c}p+1]} - \frac{p(1-e^{-ap})}{[p^2+D^2][p^2+2\bar{c}p+1]}\right\} \quad (82)$$

$$= \frac{\delta_1}{y_c''} - \frac{\delta_2}{y_c''} \qquad \text{say}$$

where

$$\frac{\delta_1}{y_c''} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1-e^{-ap}}{p(p^2+2\bar{c}p+1)}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{-\bar{c}\tau}}{\eta}\sin\left[\eta\tau-\psi_1\right]+1\right\}$$

$$-\frac{1}{2}\left\{1+\frac{e^{-\bar{c}[\tau-a]}}{\eta}\sin\left(\eta[\tau-a]-\psi_1\right)H\left(\tau-a\right)\right\} \quad (83)$$

and

$$\frac{\delta_2}{y_c''} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{p(1-e^{-ap})}{(p^2+D^2)(p^2+2\bar{c}p+1)}\right\}$$

$$= \frac{1}{2}\frac{D^2}{\sqrt{(D^2-1)^2+4\bar{c}^2 D^2}}\left\{\cos\left(D\tau-\psi_2\right)\right.$$

$$+ \frac{e^{-\bar{c}\tau}}{\eta}\sin\left(\eta\tau-\psi_3\right) - \cos\left(D[\tau-a]-\psi_2\right)H(\tau-a) - \frac{e^{-\bar{c}[\tau-a]}}{\eta}\sin\left(\eta[\tau-a]-\psi_3\right)H(\tau-a)$$

$$(84)$$

The phase angles are

$$
\begin{aligned}
\gamma_1 &= \arctan\left(\frac{\eta}{-\bar{z}}\right) \\[2mm]
\gamma_2 &= \arctan\left(\frac{2\bar{z}\,D}{D^2-1}\right) \\[2mm]
\gamma_3 &= -\gamma_1 - \arctan\left(\frac{2\eta\bar{z}}{D^2+2\bar{z}-1}\right)
\end{aligned}
\right\} \tag{85}
$$

The complete general solution of Eq. (82) is thus

$$
\begin{aligned}
\frac{\delta}{Y_c''} &= \frac{e^{-\bar{z}\tau}}{2\eta}\sin\left(\eta\tau-\gamma_1\right) + \frac{1}{2} \\[2mm]
&\quad - \left\{\frac{1}{2} + \frac{e^{-\bar{z}[\tau-a]}}{2\eta}\sin\left(\eta[\tau-a]-\gamma_1\right)\right\} H(\tau-a) \\[2mm]
&\quad - \frac{D^2}{\sqrt{(D^2-1)^2+4\bar{z}^2 D^2}}\,\frac{1}{2}\left\{\cos\left(D\tau-\gamma_2\right)+\frac{e^{-\bar{z}\tau}}{\eta}\sin(\eta\tau-\gamma_3)\right. \\[2mm]
&\quad - \cos\left(D[\tau-a]-\gamma_2\right)H(\tau-a) \\[2mm]
&\quad \left. - \frac{e^{-\bar{z}[\tau-a]}}{\eta}\sin\left(\eta[\tau-a]-\gamma_3\right)H(\tau-a)\right\} \tag{86}
\end{aligned}
$$

We now separate the general equation into the two time regimes, $\tau < a$ and $\tau > a$

(i) $\underline{\tau < a}$

$$
H(\tau-a) = 0
$$

so that eq. (86) reduces to

$$
\begin{aligned}
\frac{\delta}{Y_c''} &= \frac{e^{-\bar{z}\tau}}{2\eta}\sin\left(\eta\tau-\gamma_1\right) + \frac{1}{2} \\[2mm]
&\quad - \frac{D^2}{\sqrt{(D^2-1)^2-4\bar{z}^2 D^2}}\,\frac{1}{2}\left\{\cos\left(D\tau-\gamma_2\right)+\frac{e^{-\bar{z}\tau}}{\eta}\sin\left(\eta\tau-\gamma_3\right)\right\}
\end{aligned}
$$

$$\tag{87}$$

Differentiating,

$$\frac{\delta'}{Y_c''} = \frac{e^{-\bar{c}\tau}}{2}\left\{-\frac{\bar{c}}{\eta}\sin(\eta\tau-\gamma_1) + \cos(\eta\tau-\gamma_1)\right\}$$

$$-\frac{D^2/2}{\sqrt{(D^2-1)^2+4\bar{c}^2D^2}}\left\{-D\sin(D\tau-\gamma_2)\right.$$

$$\left. + e^{-\bar{c}\tau}\left[-\frac{\bar{c}}{\eta}\sin(\eta\tau-\gamma_3) + \cos(\eta\tau-\gamma_3)\right]\right\}$$

$$= \frac{e^{-\bar{c}\tau}}{2\eta}\sin\eta\tau$$

$$+\frac{D^2/2}{\sqrt{(D^2-1)^2+4\bar{c}^2D^2}}\left\{D\sin(D\tau-\gamma_2) + \frac{e^{-\bar{c}\tau}}{\eta}\sin(\eta\tau-\gamma_3+\gamma_1)\right\} \tag{88}$$

$$= 0 \text{ when } \tau=\tau^* \text{ and } \delta=\delta_{max}$$

A graphical method of solution was used to solve for τ^*, and δ_{max} found by substituting $\tau=\tau^*$ into eq. (87).

(ii) $\underline{\tau > a}$

For this case, $H(\tau-a) = 1$, and eq. (86) becomes

$$\frac{\delta}{Y_c''} = \frac{e^{-\bar{c}\tau}}{2\eta}\sin(\eta\tau-\gamma_1) - e^{-\bar{c}[\tau-a]}\sin(\eta[\tau-a]-\gamma_1)$$

$$-\frac{D^2/2}{\sqrt{(D^2-1)^2+4\bar{c}^2D^2}}\left\{e^{-\bar{c}\tau}\sin(\eta\tau-\gamma_3)\right.$$

$$\left. + \frac{e^{-\bar{c}[\tau-a]}}{\eta}\sin(\eta[\tau-a]-\gamma_3)\right\} \tag{89}$$

Differentiating $\frac{\delta}{Y_c''}$ with respect to τ and simplifying the result gives the following result:

35

$$\frac{\delta'}{Y_c''} = \frac{e^{-\bar{c}\tau}}{2\eta}\left\{\sin\eta\tau - \frac{D^2}{\sqrt{(D^2-1)^2+4\bar{c}^2 D^2}}\sin\left(\eta\tau - \gamma_3 + \gamma_1\right)\right\}$$

$$- \frac{e^{-\bar{c}[\tau-a]}}{2\eta}\left\{\sin\eta[\tau-a] + \frac{D^2}{\sqrt{(D^2-1)^2+4\bar{c}^2 D^2}}\sin\left(\eta[\tau-a] - \gamma_3 + \gamma_1\right)\right\}$$

Equating to zero, and solving for $\tau = \tau^*$ gives

$$\tau^* = \frac{1}{\eta}\arctan\left\{-\frac{\left(J_1\sin\Theta_1 - J_2\sin\Theta_2\right)}{J_1\cos\Theta_1 - J_2\cos\Theta_2}\right\} \tag{90}$$

where

$$
\begin{aligned}
J_1 &= \sqrt{1 - \frac{D^2}{\sqrt{(D^2-1)^2+4\bar{c}^2 D^2}}\cos\left(\gamma_1 - \gamma_3\right)} \\[2mm]
J_2 &= \sqrt{1 + \frac{D^2}{\sqrt{(D^2-1)^2+4\bar{c}^2 D^2}}\cos\left(\gamma_1 - \gamma_3\right)} \\[2mm]
\Theta_1 &= \arctan\left\{\frac{D^2}{J_1\sqrt{(D^2-1)^2+4\bar{c}^2 D^2}}\sin\left(\gamma_1 - \gamma_3\right)\right\} \\[2mm]
\Theta_2 &= \arctan\left\{\frac{D^2}{J_2\sqrt{(D^2-1)^2+4\bar{c}^2 D^2}}\sin\left(\gamma_1 - \gamma_3\right)\right\}
\end{aligned}
\right\} \tag{91}
$$

$\dfrac{\delta_{max}}{Y_c''}$ is found by substituting equations (90) and (91) into eq. (89).

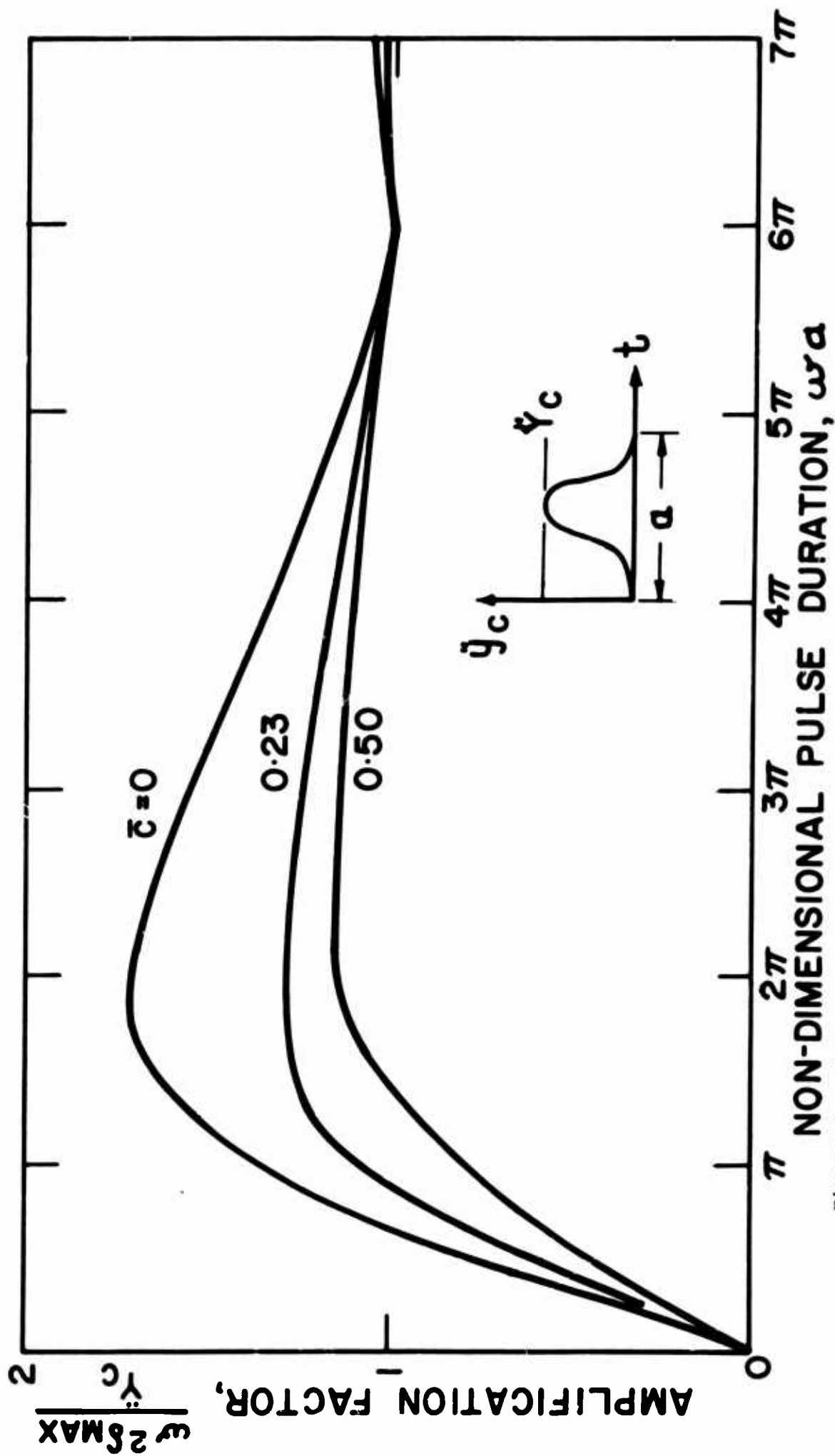The response of the system to a versed-sine pulse is plotted in Figure 8.

Figure 8. Response of the System to a Versed-sine Pulse (Zero Initial Conditions).